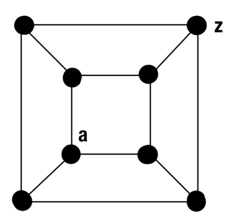
Exam Stochastic Processes

18 June 2021

- You have from 08.30 until 12.00. This includes the time needed to take pictures of your work and upload it to nestor dropbox.
- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- During the entire time you should be connected to the video call and be on camera, with sound turned on. Failure to do this will count as cheating.

Exercise 1 (20 pts)
Consider the Galton-Watson process with offspring distribution given by $\mathbb{P}(X = k) = (1-p)p^k$ for $k = 0, 1, 2, \dots$. As usual, Z_t denotes the number of individuals in generation t and $Z_{\text{tot}} := Z_0 + Z_1 + \dots$ the total number of all individuals in all generations together.
Determine $\mathbb{P}(Z_1 = k | Z_{\text{tot}} < \infty)$ for every $k \in \mathbb{Z}_{\geq 0}$.
Clearly justify all your steps and state any results from the lecture notes or tutorial sheets you are using.

Exercise 2 (20 pts)
Consider the following network, where all conductances are one and a, z are as indicated.



Determine the effective resistance and the escape probability.
Make sure to clearly justify all your steps.

(*Hint: What can you say about the voltages of the neighbours of* a*? And what about the voltages of the neighbours of* z*? It may help draw the network differently.*)

Exercise 3 (20 pts)
Let X, Y, U, V be independent random variables taking values in $\mathbb{Z}$. Show that

$$d_{\text{TV}}(X + Y, U + V) \leq d_{\text{TV}}(X, U) + d_{\text{TV}}(Y, V).$$

(see next page)

Exercise 4 (20 pts)

We consider the following variant of Pólya's urn model. Initially there are r red balls and b blue balls in the urn. We repeatedly grab a ball uniformly at random from the urn, put it back and add c balls of the same colour. Let R_n denote the number of red balls after the n-th time we've done this, and let

$$X_n := \frac{R_n}{c \cdot n + r + b},$$

denote the corresponding fraction of red balls.

Show that $(X_n)_n$ is a martingale (wrt. itself).

Exercise 5 (20 pts)

You want to cross a busy street. The times when the cars pass in front of you behave like a Poisson process of intensity λ, and in order to cross you will need an interval of time of length $\geq a$ during which no cars pass.

Let T be the (random) time from now until you will have managed to cross the street. Observe for instance that $\mathbb{P}(T = a) = \mathbb{P}(\text{no cars in } [0, a]) = e^{-\lambda a}$.

Show that $\mathbb{E}T = \frac{1}{\lambda} \cdot (e^{\lambda a} - 1)$.

(*Hint: It helps to use the formulation of the Poisson process in terms of interarrival times. What distribution does the number N of cars that pass until you cross have? What is the expected time you wait until the first car, given that you cannot pass before the first car? What is $\mathbb{E}(T|N = n)$?*)

the end